Pointwise error estimates for a generalized Oseen problem and an application to an optimal control problem☆

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Abstract

We derive pointwise error estimates for a generalized Oseen problem when it is approximated by an low order Taylor–Hood finite element scheme in two dimensions. The analysis is based on estimates for regularized Green's functions associated with a generalized Oseen problem on weighted Sobolev spaces and weighted interpolation results. We apply the maximum norm results to obtain convergence in an optimal control problem governed by a generalized Oseen equation, and present a numerical example that allows us to show the behaviour of the error.

Keywords: Pointwise error estimates, Generalized Oseen, Optimal Control.

Introduction

The approximation of the solution for optimal control problems governed by partial differential equations (see [1]), by means of finite element schemes has been an active research area in the last several years. In this work we are mainly interested in the approximation of the quadratic optimal control problem when the state equation is given by a generalized Oseen problem without any control constraint, and the objective is to reduce the pressure of the fluid flow in certain points of a given domain, where the state equation is posed.

In this context, we investigate the resolution and convergence of the following quadratic optimal control problem:

$$\inf_{z_c \in U_{ad}} \mathcal{J}(z_c) \quad \text{with} \quad \mathcal{J}(u_c) = \frac{1}{2}\|z_o - z_d\|^2_{\mathbb{R}^M} + \frac{\eta}{2}\|z_c\|^2_{\mathcal{U}},$$

where the main goal is to take the observation $z_o = z_o(p(z_c))$ of the state of the system $p(z_c)$, the fluid pressure, which follows the generalized Oseen dynamic, as close as possible to a desired state z_d, in a set of M given points in the fluid domain, through the application of a control variable z_c, living in a set of admissible controls $U_{ad} \subset \mathcal{U}$, which entails a cost in the control process with η denoting the weighting factor in the cost functional $\mathcal{J}(z_c)$.

Several authors have proposed and analyzed different techniques for solving optimal control problems (see [2] and references therein), but from all the different

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choices we will consider the one known as the formulation of the reduced optimization problem, where finite element schemes are used to approximate the solution of the state equation, the space of admissible controls is discretized together with the cost functional leading to the resolution of a quadratic programming problem. The reason for such a choice is that the optimal control strategy does not require any information about the related adjoint problem and its discretization. This scheme is analyzed in [3], where a key result is that the convergence for the optimal control variable is governed by the convergence of the finite element scheme of the state equation. Thus, since our main objective is to reduce the pressure of the fluid flow in certain points, maximum-norm stability results for the state are crucial to obtain convergence for the control variable (see, for instance, [3, 4, 5] where this methodology is used). We note that, the reason for the choice of such dynamics in the state is due to the fact that the treatment for solving the time dependent Navier–Stokes equations is based on the resolution of the generalized Oseen problem.

Pointwise error estimates go back to [6, 7] and in [8] an overview and recent improvements on pointwise error estimates for the Poisson problem can be found. Similar error estimates were derived for fluid flow problems such as the Stokes equation by using weighted inf-sup conditions in [9, 10] or by using local projection operators in [11, 12, 13]. We refer to these cited references for a more comprehensive overview of previous literature on maximum norm results for the Stokes problem.

In this manuscript, a quasi optimal convergence is attained for pointwise error estimates when a generalized Oseen problem is approximated by using the lowest order Taylor–Hood finite element. Our proof is based on the technique used in [9] (see also [10]), where the results were derived by using weighted inf-sup conditions or a global weighted technique. These are standard approaches for obtaining error estimates in the maximum norm for finite element methods. The proofs of these error estimates are based on the estimates in weighted Sobolev norms for some so-called regularized Green's functions and their finite element approximations. In [9] a weighted norm with the weight function introduced by Natterer [6] is used, and [10] has improved the result by removing the logarithmic factor by working with another weight function (see [8]). In both cases, the special behaviour of the discrete pressure should be taken into account, and a complete analysis of the regularity of the solution is needed. Here, due to the presence of the advective term in all the analysis of the generalized Oseen equation, we will follow the first approach presented in [9].

We note that, there is a modern technique contained in the recent works [13] and [14], which also remove the logarithmic terms for quadratic Taylor–Hood elements for the Stokes problem. Unfortunately, it is not clear how to apply this technique to remove the logaritmic term for the Oseen equation, due to the presence of the advective term which has to be taken into account through all of the error analysis. Such a problem is avoided in the case of the Navier–Stokes problem, since the usual smallness assumptions are used.

The manuscript is organized as follows. In Section 1 we present the generalized Oseen problem, its weak formulation and the finite element scheme with classical convergence results. Also, we introduce weighted norms in Sobolev spaces and interpolation results which are vital for the analysis. In Section 2 we present the main result of pointwise convergence and provide the main steps for the error analysis. We note that, there will be many similarities between the proofs in this manuscript and the proofs from [9], nevertheless, in order to make our paper self contained we present a detailed analysis. We devote Section 3 to prove some auxiliary results needed in the convergence analysis. In Section 4 we present the optimal control problem, the resolution strategy and the convergence result for the optimal control problem.

1. Preliminaries

We shall use standard notation for Sobolev and Lebesgue spaces, norms, and inner products. Namely, for a bounded open domain, $G \subseteq \mathbb{R}^2$: $L^2(G)$ denotes the space of square integrable functions over G, $L^2_0(G)$ represents functions belonging to $L^2(G)$ with zero average in G, $H^1(G)$ is the usual Sobolev space and $H^1_0(G)$ denotes the subspace of $H^1(G)$ consisting of functions whose trace is zero on the boundary of G. Let $\langle \cdot, \cdot \rangle_G$ denote the inner product in $L^2(G)$ (or in $L^2(G)^2$ or in $L^2(G)^{2\times 2}$ when necessary). The norm of the space $H^m(G)$ is denoted by $\| \cdot \|_{H^m(G)}$ and the norm of the Lebesgue space $L^2(G)$ is denoted by $\| \cdot \|_{L^2(G)}$ and we use bold letters to denote the vector-valued counterparts of the Sobolev and Lebesgue spaces, e.g., $\boldsymbol{H}^1_0(G) = H^1_0(G)^2$.

1.1. Model problem: state equation

We are interested in the study of the following generalized Oseen problem: *Find $\boldsymbol{u}$ and p such that*

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{and} \quad \mathbf{div}\ \boldsymbol{u} = 0 \text{ in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\boldsymbol{u}$ represents the fluid velocity and p its pressure, > 0 is the fluid viscosity and the problem data is taken such that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{b} \in \boldsymbol{W}^{1,\infty}(\Omega)$ is a solenoidal vector field ($\mathbf{div}\ \boldsymbol{b} = 0$), and Ω is a bounded domain in $\mathbb{R}^2$.

The weak formulation of (1) reads as follows: *Find $(\boldsymbol{u}, p) \in \boldsymbol{H}^1_0(\Omega) \times L^2_0(\Omega)$ such that*

$$\begin{cases} \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{b} \cdot \nabla \boldsymbol{u} + \boldsymbol{u}, \boldsymbol{v} \rangle_\Omega - \langle p, \mathbf{div}\ \boldsymbol{v} \rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega \quad \forall\ \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega), \\ \langle q, \mathbf{div}\ \boldsymbol{u} \rangle_\Omega = 0 \quad \forall\ q \in L^2_0(\Omega). \end{cases} \tag{2}$$

The well-posedness of (2) follows from the fact that the bilinear form $\langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v} \rangle_\Omega$ is coercive on the space $\boldsymbol{V}_0 = \{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) : \ \mathbf{div}\ \boldsymbol{v} = 0\}$ owing to the Poincare's inequality and there exists a constant $\beta > 0$ such that

$$\sup_{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \setminus \{\boldsymbol{0}\}} \frac{\langle q, \mathbf{div}\ \boldsymbol{v} \rangle_\Omega}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1_0(\Omega)}} \geq \beta \|q\|_{L^2(\Omega)} \qquad \forall\ q \in L^2_0(\Omega).$$

See Chapter 4 in [15] or Chapter 4 in [16] for more details.

As we stated in the introduction we will mainly focus on the approximation of (2) by the lowest order Taylor–Hood approximation, that is approximate the velocity field by means of piecewise continuous polynomials of degree two and the pressure by piecewise continuous polynomials of degree one ($\mathbb{P}^2_2 - \mathbb{P}_1$). From now on, we consider shape-regular and quasi-uniform family of triangulations $\{\mathcal{P}_h\}$ of the domain Ω, then based on a given partition, a discrete finite element approximation reads: *Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that*

$$\begin{cases} \langle \nabla \boldsymbol{u}_h, \nabla \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{b} \cdot \nabla \boldsymbol{u}_h + \boldsymbol{u}_h, \boldsymbol{v} \rangle_\Omega - \langle p_h, \mathbf{div}\ \boldsymbol{v} \rangle_\Omega = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}_h, \\ \langle q, \mathbf{div}\ \boldsymbol{u}_h \rangle_\Omega = 0 \quad \forall\ q \in Q_h, \end{cases} \tag{3}$$

where $\boldsymbol{V}_h \subset \boldsymbol{H}^1_0(\Omega)$ and $Q_h \subset L^2_0(\Omega)$. The existence of the discrete solution and its stability follows now from the classical discrete inf-sup condition

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_h \setminus \{\boldsymbol{0}\}} \frac{\langle q, \mathbf{div}\ \boldsymbol{v} \rangle_\Omega}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1_0(\Omega)}} \geq \beta \|q\|_{L^2(\Omega)} \qquad \forall q \in Q_h, \tag{4}$$

where β is a positive constant independent of the mesh size. Now, if (4) is satisfied then optimal error estimates can be obtained, namely

$$\begin{aligned} & \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}_0^1(\Omega)} + \|p-p_h\|_{L^2(\Omega)} \\ & \quad \leq C\left\{\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)} + \inf_{q\in Q_h}\|p-q\|_{L^2(\Omega)}\right\}, \end{aligned} \tag{5}$$

and under some regularity assumption on the domain,

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \leq \quad Ch\left\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}_0^1(\Omega)} + \|p-p_h\|_{L^2(\Omega)}\right\}. \tag{6}$$

Modifying the mass equation in (1) by taking $\mathbf{div}\ \boldsymbol{u} = g$ in Ω, allowing $g \in H_0^1(\Omega)$ with Ω being a convex polygon in $\mathbb{R}^2$, then

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C\left\{\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla g\|_{L^2(\Omega)}\right\}. \tag{7}$$

See [16, 17, 18] for more details of the above results. We will henceforth assume that Ω is a convex polygon in $\mathbb{R}^2$.

Remark 1. *We have to mention that the previous convergence results, do not need a quasi-uniform partition, only a shape-regular family of affine simplices are sufficient, but in order to develop the subsequent analysis will be a fundamental requirement, since is related with some properties concerning weighted Sobolev spaces.*

1.2. Weighted norms in Sobolev spaces and interpolation operators

In order to define weighted norms, we take the following function as the weight

$$\sigma_i(\boldsymbol{x}) := \left(|\boldsymbol{x}-\boldsymbol{x}_i|^2 + \theta^2\right)^{1/2} \quad \text{where} \quad \boldsymbol{x}, \boldsymbol{x}_i \in \Omega \quad \text{and} \quad \theta = Kh, \tag{8}$$

where K is a positive constant to be specified later on (see Lemmas 4 and 5), and $\boldsymbol{x}_i$ is a point where a value close to the maximum will be attained. From [19] it follows that for $1 \leq i \leq 3$,

$$\left\{\begin{array}{c} \max_{\boldsymbol{x}\in T}\sigma_i(\boldsymbol{x}) \leq C\min_{\boldsymbol{x}\in T}\sigma_i(\boldsymbol{x}) \text{ for all } T \in \mathcal{P}_h, \\ |D^j\sigma_i^{\alpha}(\boldsymbol{x})| \leq C(j,\alpha)\sigma_i^{\alpha-j}(\boldsymbol{x}), \end{array}\right. \tag{9}$$

where $\alpha \in \mathbb{R}$ and $D^j f$ denotes the tensor of derivatives of order j of f, and using a transformation to polar coordinates it is possible to conclude that

$$\int_\Omega \sigma_i^{-2}(k\boldsymbol{x})d\boldsymbol{x} \quad \leq \quad C|\log\theta|, \quad \forall\ k > 0, \tag{10}$$

and θ small enough. For $\alpha \in \mathbb{R}$ and j a non-negative integer, weighted semi-norms are defined by

$$\|D^j q\|^2_{\sigma_i^\alpha} := \sum_{|\beta|=j}\int_\Omega |\partial^\beta q|^2\sigma_i^\alpha\ d\boldsymbol{x}, \qquad \forall q \in H^j(\Omega).$$

In all the error analysis a crucial factor is to have at hand some interpolation results. First we remark that an equivalent result to (4) (see Lemma 4.19 in [16]), is the existence of an operator, $\Pi_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h$ which satisfies

$$\|\Pi_h\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{H}_0^1(\Omega)}, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{11}$$

$$\langle \mathbf{div}(\boldsymbol{v}-\Pi_h\boldsymbol{v}), q\rangle_\Omega = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \text{ and } \forall q \in Q_h, \tag{12}$$

usually call the Fortin operator. Such an operator has been constructed locally for our discrete scheme in [20, 21], and it satisfies for $\alpha \in \mathbb{R}$

$$\|D^j(\boldsymbol{v} - \Pi_h \boldsymbol{v})\|_{\sigma_i^\alpha} \le Ch^{2-j}\|D^2\boldsymbol{v}\|_{\sigma_i^\alpha}, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^2(\Omega), \quad j = 0, 1 \tag{13}$$

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} \le Ch\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)}, \qquad \forall \boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega), \tag{14}$$

$$\|\nabla \Pi_h \boldsymbol{v}\|_{\sigma_i^2} \le C\|\nabla \boldsymbol{v}\|_{\sigma_i^2}, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{15}$$

$$\|\nabla(\sigma^2 \boldsymbol{v} - \Pi_h(\sigma_i^2 \boldsymbol{v}))\|_{\sigma_i^{-2}} \le Ch\|\boldsymbol{v}\|_{\sigma_i^{-2}} + Ch\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{16}$$

and taking P_h as the Scott–Zhang interpolator (see [22, 16]), it follows that

$$\|q - P_h q\|_{\sigma_i^\alpha} \le Ch\|Dq\|_{\sigma_i^\alpha}, \qquad \forall q \in H^1(\Omega), \tag{17}$$

$$\|\sigma_i^2 q - P_h(\sigma_i^2 q)\|_{\sigma_i^{-2}} \le Ch\|q\|_{L^2(\Omega)}, \qquad \forall q \in Q_h. \tag{18}$$

Remark 2. *We have to mention that the introduction to three different weights, is due to the fact that the error analysis involves three regularize Green's functions in order to study the velocity field, its gradient and pressure field. Now, the points $\boldsymbol{x}_i$ $(i = 1, 2, 3)$, are such that maximum of related quantities to those problems will be atained, for example $\boldsymbol{x}_1$ is a point where $\boldsymbol{u}_h - \Pi_h \boldsymbol{u}$ atained its maximum value, with Π_h a suitable projection operator.*

1.3. Regularization of the Dirac mass

In order to study the error in the velocity and the pressure approximation we will define generalized Oseen problem with different right hand sides in the momentum and mass conservation equation using an appropriate mollifier δ_M, which is a suitable regularization of the Dirac delta measure. To introduce the mollifier, let $\boldsymbol{x}_i$, for $1 \le i \le 3$, be a fixed point in Ω and let $T_i \in \mathcal{P}_h$ be such that $\boldsymbol{x}_i$ lies in the interior of T_i, then take the mollifier $\delta_{M,i} \in \mathcal{C}_0^\infty(\Omega)$ satisfying

$$\text{supp } \delta_{M,i} \subset B_i, \tag{19}$$

$$\int_\Omega \delta_{M,i} d\boldsymbol{x} = 1 \quad \text{with} \quad \delta_{M,i} \ge 0, \tag{20}$$

$$\|D^j \delta_{M,i}\|_{L^\infty(\Omega)} \le Ch^{-2-j} \quad \text{for } j = 0, 1, \tag{21}$$

where B_i is a ball of radius αh contained in T_i and $\alpha > 0$ is a suitable constant. Now, let χ be any piecewise polynomial (without inter-element continuity requirements), and assume that its maximum norm is attained at $\boldsymbol{x}_i$ and χ is extended to the closure of T_i by continuity. Then, using the shape-regular property of the mesh, it is always possible to construct a ball B_i with center $\boldsymbol{y}_i$ so that $|\boldsymbol{x}_i - \boldsymbol{y}_i| = C\alpha h$, and it is also possible to prove that

$$\int_\Omega \chi \delta_{M,i} \, d\boldsymbol{x} = \chi(\boldsymbol{z}) \quad \text{for some } \boldsymbol{z} \in B_i \quad \text{and} \quad \|\chi\|_{L^\infty(\Omega)} \le 2 \left| \int_\Omega \chi \delta_{M,i} \, d\boldsymbol{x} \right|. \tag{22}$$

See [9] for more details.

2. Pointwise error estimation

The main result of our work is to establish L^∞ error estimates when the velocity and pressure fields of the generilized Oseen problem (1) are approximated by the lowest order Taylor–Hood scheme. This will be established in the following theorem.

Theorem 1. *Let Ω be a convex polygon in $\mathbb{R}^2$, $(\boldsymbol{u}, p)$ the solution of* (2) *and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ the solution of* (3). *If $\boldsymbol{u} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and $p \in L^\infty(\Omega)$, then there exists a constant $C > 0$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^\infty(\Omega)} \leq Ch|\log h| \left(\|\nabla(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{\boldsymbol{L}^\infty(\Omega)} + \|p - P_h p\|_{L^\infty(\Omega)} \right), \quad (23)$$

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{\boldsymbol{L}^\infty(\Omega)} \leq C|\log h|^{\frac{1}{2}} \left(\|\nabla(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{\boldsymbol{L}^\infty(\Omega)} + \|p - P_h p\|_{L^\infty(\Omega)} \right), \quad (24)$$

$$\|p - p_h\|_{L^\infty(\Omega)} \leq C|\log h|^{\frac{1}{2}} \left(\|\nabla(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{\boldsymbol{L}^\infty(\Omega)} + \|p - P_h p\|_{L^\infty(\Omega)} \right). \quad (25)$$

The proof of the previous theorem is based on the study of three regularized Green's functions associated to the Generalized Oseen problem with different right hand sides. These are summarized in the next three lemmas, whose proofs are very technical and, for better understanding of the reader, are given in the next section.

Velocity field analysis: The regularized Green's function is as follows

$$\left\{ \begin{array}{c} -\Delta \boldsymbol{G}_1 + \boldsymbol{b} \cdot \nabla \boldsymbol{G}_1 + \boldsymbol{G}_1 + \nabla \lambda_1 = \boldsymbol{\delta}_{M,1} \quad \text{and} \quad \textbf{div } \boldsymbol{G}_1 = 0 \quad \text{in } \Omega, \\ \boldsymbol{G}_1 = \boldsymbol{0} \quad \text{on } \partial\Omega, \end{array} \right. \quad (26)$$

where $\boldsymbol{\delta}_{M,1}$ can be $(\delta_{M,1}, 0)$ or $(0, \delta_{M,1})$. We have the following result for the previous problem.

Lemma 1. *For the solution of* (26), *there exists a constant $C > 0$ such that*

$$\|\boldsymbol{G}_1\|_{\sigma_1^2} + \|D^2 \boldsymbol{G}_1\|_{\sigma_1^2} + \|\nabla \boldsymbol{G}_1\|_{\sigma_1^2} + \|\nabla \lambda_1\|_{\sigma_1^2} \leq C|\log h|^{1/2}.$$

Gradient of the velocity field analysis: The regularized Green's function is as follows

$$\left\{ \begin{array}{c} -\Delta \boldsymbol{G}_2 + \boldsymbol{b} \cdot \nabla \boldsymbol{G}_2 + \boldsymbol{G}_2 + \nabla \lambda_2 = D\boldsymbol{\delta}_{M,2} \quad \text{and} \quad \textbf{div } \boldsymbol{G}_2 = 0 \text{ in } \Omega, \\ \boldsymbol{G}_2 = \boldsymbol{0} \quad \text{on } \partial\Omega, \end{array} \right. \quad (27)$$

where $D\boldsymbol{\delta}_{M,2}$ stands for any directional derivative of either $(\delta_{M,2}, 0)$ or $(0, \delta_{M,2})$. Associated with this problem we have the following result.

Lemma 2. *For problem* (27), *there exists a constant $C > 0$ such that*

$$\|\boldsymbol{G}_2\|_{\sigma_2^2} + \|D^2 \boldsymbol{G}_2\|_{\sigma_2^2} + \|\nabla \boldsymbol{G}_2\|_{\sigma_2^2} + \|\nabla \lambda_2\|_{\sigma_2^2} \leq Ch^{-1}.$$

Pressure field analysis: Taking $\phi \in C_0^\infty(\Omega)$ satisfying $\langle \phi, 1 \rangle_\Omega = 1$, the regularized Green's function is as follows

$$\left\{ \begin{array}{c} -\Delta \boldsymbol{G}_3 + \boldsymbol{b} \cdot \nabla \boldsymbol{G}_3 + \nabla \lambda_3 + \boldsymbol{G}_3 = \boldsymbol{0} \quad \text{and} \quad \textbf{div } \boldsymbol{G}_3 = \delta_{M,3} - \phi \quad \text{in } \Omega, \\ \boldsymbol{G}_3 = \boldsymbol{0} \quad \text{on } \partial\Omega. \end{array} \right. \quad (28)$$

The solution of this problem satisfies the following result.

Lemma 3. *For problem* (28), *there exists a constant $C > 0$ such that*

$$\|\boldsymbol{G}_3\|_{\sigma_3^2} + \|D^2 \boldsymbol{G}_3\|_{\sigma_3^2} + \|\nabla \boldsymbol{G}_3\|_{\sigma_3^2} + \|\nabla \delta_{M,3}\|_{\sigma_3^2} \leq Ch^{-1}.$$

Proof. Similarly to the prove of the previous Lemmas, it follows that

$$\begin{array}{c} \|\mu_j \boldsymbol{G_3}\|_{\boldsymbol{L}^2(\Omega)} + \|D^2(\mu_j \boldsymbol{G_3})\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\mu_j \boldsymbol{G_3})\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\mu_j \lambda_3)\|_{\boldsymbol{L}^2(\Omega)} \\ \leq C\{\|\nabla \boldsymbol{G_3}\|_{\boldsymbol{L}^2(\Omega)} + \|\lambda_3\|_{L^2(\Omega)}\}, \end{array}$$

and by using (19) and (21), yields that $\|\nabla \boldsymbol{G_3}\|_{\boldsymbol{L}^2(\Omega)} + \|\lambda_3\|_{L^2(\Omega)} \leq C\|\delta_3 - \phi\|_{L^2(\Omega)} \leq Ch^{-1}$. The remaing terms can be bounded by using the inf-sup condition (4).

We are now led to estimate the terms in norm Another result that we will use in the error analysis related to all the auxiliary problems (26), (27) and (28) is the next one. Again, the proof will be included in the next section.

Theorem 2. *For $\Omega \subset \mathbb{R}^2$ being a convex polygon, letting $(\boldsymbol{G}_i, \lambda_i)$ be the solution of any of the regularized problems (26), (27) or (28), and $(\boldsymbol{G}_i^h, \lambda_i^h) \in \boldsymbol{V}_h \times Q_h$ its discrete approximations (with $i = 1, 2, 3$). Then, for K large enough, there exists a constant $C = C(K)$ such that*

$$\begin{aligned}\|\boldsymbol{G}_i - \boldsymbol{G}_i^h\|_{\sigma^2} + \|\nabla(\boldsymbol{G}_i - \boldsymbol{G}_i^h)\|_{\sigma^2} \quad &\leq \quad Ch\Big(\{\|D^2\boldsymbol{G}_i\|_{\sigma^2} + \|\nabla\lambda\|_{\sigma^2}\} \\ &\qquad + h\{\|D^2\boldsymbol{G}_i\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\lambda_i\|_{L^2(\Omega)}\}\Big),\end{aligned}$$

where K is given in (8), and σ is the corresponding weight.

We will prove Theorem 1 in three stages to obtain (23), (24) and (25), respectively, and in some cases we will drop the sub index i when no confusion arises.

PROOF. **Estimation for the Velocity field** (23)**:** From (2) and (3), we have the following Galerkin orthogonality, for all $(\boldsymbol{v}, q) \in (\boldsymbol{V}_h \times Q_h)$:

$$\left\{\begin{aligned}\langle\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \nabla\boldsymbol{v}\rangle_\Omega + \langle\boldsymbol{b}\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_h) + \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}\rangle_\Omega & \\ - \langle p - p_h, \mathbf{div}\ \boldsymbol{v}\rangle_\Omega &= 0, \\ \langle q, \mathbf{div}\ (\boldsymbol{u}-\boldsymbol{u}_h)\rangle_\Omega &= 0.\end{aligned}\right. \tag{29}$$

Similarly, we also have an error equation for the regularized Green's function (26), $(\boldsymbol{v}, q) \in (\boldsymbol{V}_h \times Q_h)$:

$$\left\{\begin{aligned}\left\langle\nabla\left(\boldsymbol{G}_1 - \boldsymbol{G}_1^h\right), \nabla\boldsymbol{v}\right\rangle_\Omega + \left\langle\boldsymbol{b}\cdot\nabla(\boldsymbol{G}_1 - \boldsymbol{G}_1^h) + \boldsymbol{G}_1 - \boldsymbol{G}_1^h, \boldsymbol{v}\right\rangle_\Omega & \\ - \left\langle\lambda_1 - \lambda_1^h, \mathbf{div}\ \boldsymbol{v}\right\rangle_\Omega &= 0, \\ \left\langle q, \mathbf{div}\ \left(\boldsymbol{G}_1 - \boldsymbol{G}_1^h\right)\right\rangle_\Omega &= 0.\end{aligned}\right. \tag{30}$$

We start the analysis studying $\Pi_h\boldsymbol{u} - \boldsymbol{u}_h$. For this, taking $\boldsymbol{x}_1 \in \Omega$ to be the point where the maximum is attained in $\|\Pi_h\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^\infty(\Omega)} = \max\limits_{1\leq i\leq 2} \|u_h^i - (\Pi_h u)^i\|_{\boldsymbol{L}^\infty(\Omega)}$, then using (22) with the weak form of (26) tested with $\boldsymbol{v} = \Pi_h\boldsymbol{u} - \boldsymbol{u}_h$ and (22), it follows that

$$\begin{aligned}\|\Pi_h\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^\infty(\Omega)} &\leq 2|\langle\Pi_h\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\delta}_{M,1}\rangle_\Omega| \\ &= 2\Big|\left\langle\nabla\boldsymbol{G}_1^h, \nabla(\Pi_h\boldsymbol{u} - \boldsymbol{u}_h)\right\rangle_\Omega + \left\langle\boldsymbol{b}\cdot\nabla\boldsymbol{G}_1^h, \Pi_h\boldsymbol{u} - \boldsymbol{u}_h\right\rangle_\Omega \\ &\qquad + \left\langle\boldsymbol{G}_1^h, \Pi_h\boldsymbol{u} - \boldsymbol{u}_h\right\rangle_\Omega - \left\langle\mathbf{div}\ (\Pi_h\boldsymbol{u} - \boldsymbol{u}_h), \lambda_1^h\right\rangle_\Omega\Big|.\end{aligned} \tag{31}$$

Taking $\boldsymbol{v} = \boldsymbol{u}$ in (12) and $q = \lambda_1^h$ in (29), yields that $\left\langle\mathbf{div}\ (\Pi_h\boldsymbol{u} - \boldsymbol{u}_h), \lambda_1^h\right\rangle_\Omega = 0$ and now taking $\boldsymbol{v} = \boldsymbol{G}_1^h$ in (29) we obtain

$$\left\langle\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \nabla\boldsymbol{G}_1^h\right\rangle_\Omega + \left\langle\boldsymbol{b}\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_h) + \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{G}_1^h\right\rangle_\Omega - \left\langle p - p_h, \mathbf{div}\ \boldsymbol{G}_1^h\right\rangle_\Omega = \boldsymbol{0},$$

which allows us to rewrite the right hand side of (31) as

$$\begin{aligned}2\Big|&\left\langle\nabla\boldsymbol{G}_1^h, \nabla(\Pi_h\boldsymbol{u} - \boldsymbol{u}_h)\right\rangle_\Omega + \left\langle\boldsymbol{b}\cdot\nabla\boldsymbol{G}_1^h, \Pi_h\boldsymbol{u} - \boldsymbol{u}_h\right\rangle_\Omega + \left\langle\boldsymbol{G}_1^h, \Pi_h\boldsymbol{u} - \boldsymbol{u}_h\right\rangle_\Omega \\ &- \left\langle\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \nabla\boldsymbol{G}_1^h\right\rangle_\Omega - \left\langle\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{G}_1^h\right\rangle_\Omega - \left\langle\boldsymbol{b}\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \boldsymbol{G}_1^h\right\rangle_\Omega \\ &+ \left\langle p - p_h, \mathbf{div}\ \boldsymbol{G}_1^h\right\rangle_\Omega\Big|.\end{aligned}$$

Then, considering $q = p_h - P_h p$ in (30) we have that $\left\langle p_h - P_h p, \mathbf{div}\ (\boldsymbol{G}_1 - \boldsymbol{G}_1^h)\right\rangle_\Omega = 0$, which allows us to rewrite the previous term as

$$
\begin{aligned}
2\Big| &\left\langle \nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right), \nabla(\Pi_h \boldsymbol{u} - \boldsymbol{u})\right\rangle_\Omega + \left\langle p - P_h p, \mathbf{div}\ \left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right)\right\rangle_\Omega \\
&+ \left\langle \boldsymbol{G}_1^h - \boldsymbol{G}_1, \Pi_h \boldsymbol{u} - \boldsymbol{u}\right\rangle_\Omega + \left\langle \boldsymbol{G}_1, \Pi_h \boldsymbol{u} - \boldsymbol{u}\right\rangle_\Omega - \left\langle \Delta \boldsymbol{G}_1, \Pi_h \boldsymbol{u} - \boldsymbol{u}\right\rangle_\Omega + \\
&\left\langle \boldsymbol{b}\cdot\nabla \boldsymbol{G}_1^h, \Pi_h \boldsymbol{u} - \boldsymbol{u}_h\right\rangle_\Omega - \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{G}_1^h\right\rangle_\Omega \Big|,
\end{aligned}
$$

upon using integration by parts. Using the fact that $\langle \boldsymbol{b}\cdot\nabla \boldsymbol{v}, \boldsymbol{u}\rangle_\Omega = -\langle \boldsymbol{b}\cdot\nabla \boldsymbol{u}, \boldsymbol{v}\rangle_\Omega$ for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, it follows that

$$
\begin{aligned}
&\left\langle \boldsymbol{b}\cdot\nabla \boldsymbol{G}_1^h, \Pi_h \boldsymbol{u} - \boldsymbol{u}_h\right\rangle_\Omega - \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{G}_1^h\right\rangle_\Omega \\
&= \left\langle \boldsymbol{b}\cdot\nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right), \Pi_h \boldsymbol{u} - \boldsymbol{u}\right\rangle_\Omega + 2\left\langle \boldsymbol{b}\cdot\nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right), \boldsymbol{u} - \boldsymbol{u}_h\right\rangle_\Omega \\
&\quad + 2\left\langle \boldsymbol{b}\cdot\nabla \boldsymbol{G}_1, \boldsymbol{u} - \boldsymbol{u}_h\right\rangle_\Omega + \left\langle \boldsymbol{b}\cdot\nabla \boldsymbol{G}_1, \Pi_h \boldsymbol{u} - \boldsymbol{u}\right\rangle_\Omega,
\end{aligned}
$$

which combined with a Hölder inequality allows to conclude that

$$
\begin{aligned}
&\|\Pi_h \boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^\infty(\Omega)} \\
&\le C\Bigg\{ \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\Big(\left\|\boldsymbol{G}_1^h - \boldsymbol{G}_1\right\|_{\boldsymbol{L^1}(\Omega)} + \|\boldsymbol{G}_1\|_{\boldsymbol{L}^1(\Omega)} + \|\nabla \boldsymbol{G}_1\|_{\boldsymbol{L}^1(\Omega)} \\
&\quad + \|\Delta \boldsymbol{G}_1\|_{\boldsymbol{L}^1(\Omega)}\Big) + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L^2}(\Omega)}\left(\left\|\nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right)\right\|_{\boldsymbol{L^2}(\Omega)} + \|\nabla \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)}\right) \\
&\qquad + \left(\|\nabla(\Pi_h \boldsymbol{u} - \boldsymbol{u})\|_{\boldsymbol{L}^\infty(\Omega)} + \|p - P_h p\|_{L^\infty(\Omega)}\right) \\
&\qquad \left(\left\|\nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right)\right\|_{\boldsymbol{L^1}(\Omega)} + \|\boldsymbol{G}_1^h - \boldsymbol{G}_1\|_{\boldsymbol{L^1}(\Omega)}\right)\Bigg\},
\end{aligned}
\tag{32}
$$

hence the problem reduces to estimating $\|\boldsymbol{G}_1^h - \boldsymbol{G}_1\|_{\boldsymbol{L^1}(\Omega)}$, $\|\nabla(\boldsymbol{G}_1^h - \boldsymbol{G}_1)\|_{\boldsymbol{L^1}(\Omega)}$, $\|\Delta \boldsymbol{G}_1\|_{\boldsymbol{L}^1(\Omega)}$, $\|\nabla \boldsymbol{G}_1\|_{\boldsymbol{L}^1(\Omega)}$ and $\|\boldsymbol{G}_1\|_{\boldsymbol{L}^1(\Omega)}$. Again using a Hölder inequality and (10), we have that

$$
\left\|\boldsymbol{G}_1^h - \boldsymbol{G}_1\right\|_{\boldsymbol{L^1}(\Omega)} \le C|\log h|^{1/2}\left\|\boldsymbol{G}_1^h - \boldsymbol{G}_1\right\|_{\sigma_1^2}, \tag{33}
$$

$$
\left\|\nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right)\right\|_{\boldsymbol{L^1}(\Omega)} \le C|\log h|^{1/2}\left\|\nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right)\right\|_{\sigma_1^2}, \tag{34}
$$

hence finally the problem reduces to estimating $\|\boldsymbol{G}_1^h - \boldsymbol{G}_1\|_{\sigma_1^2}$ and $\|\nabla(\boldsymbol{G}_1^h - \boldsymbol{G}_1)\|_{\sigma_1^2}$. Such estimates are obtained using Theorem 2, from where it follows that

$$
\begin{aligned}
\left\|\boldsymbol{G}_1^h - \boldsymbol{G}_1\right\|_{\boldsymbol{L^1}(\Omega)} &\le C|\log h|^{1/2} h\Big(\left\{\left\|D^2 \boldsymbol{G}_1\right\|_{\sigma^2} + \|\nabla\lambda_1\|_{\sigma^2}\right\} \\
&\qquad + h\left\{\left\|D^2 \boldsymbol{G}_1\right\|_{\mathbf{L}^2(\Omega)} + \|\nabla\lambda_1\|_{L^2(\Omega)}\right\}\Big), \\
\left\|\nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right)\right\|_{\boldsymbol{L^1}(\Omega)} &\le C|\log h|^{1/2} h\Big(\left\{\left\|D^2 \boldsymbol{G}_1\right\|_{\sigma^2} + \|\nabla\lambda_1\|_{\sigma^2}\right\} \\
&\qquad + h\left\{\left\|D^2 \boldsymbol{G}_1\right\|_{\mathbf{L}^2(\Omega)} + \|\nabla\lambda_1\|_{L^2(\Omega)}\right\}\Big).
\end{aligned}
$$

Lemma 1 combined with the previous inequalities, yields

$$
\begin{aligned}
\left\|\nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right)\right\|_{\boldsymbol{L^1}(\Omega)} &\le C|\log h|^{1/2}\left\|\nabla\left(\boldsymbol{G}_1^h - \boldsymbol{G}_1\right)\right\|_{\sigma_1^2} &&\le C|\log h| h, \\
\|\boldsymbol{G}_1^h - \boldsymbol{G}_1\|_{\boldsymbol{L^1}(\Omega)} &\le C|\log h|^{1/2}\|\boldsymbol{G}_1^h - \boldsymbol{G}_1\|_{\sigma_1^2} &&\le C|\log h| h, \\
\|\boldsymbol{G}_1\|_{\boldsymbol{L^1}(\Omega)} &\le C|\log h|^{1/2}\|\boldsymbol{G}_1\|_{\sigma_1^2} &&\le C|\log h|, \\
\|\nabla \boldsymbol{G}_1\|_{\boldsymbol{L^1}(\Omega)} &\le C|\log h|^{1/2}\|\nabla \boldsymbol{G}_1\|_{\sigma_1^2} &&\le C|\log h|, \\
\|\Delta \boldsymbol{G}_1\|_{\boldsymbol{L^1}(\Omega)} &\le C|\log h|^{1/2}\|\Delta \boldsymbol{G}_1\|_{\sigma_1^2} &&\le C|\log h|,
\end{aligned}
$$

which together with the inf-sup condition (11) and (6), allows us to conclude that

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L^2}(\Omega)}\|\nabla(\boldsymbol{G}_1^h-\boldsymbol{G}_1)\|_{\boldsymbol{L^2}(\Omega)} = O(h^2)$$
$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L^2}(\Omega)}\|\nabla\boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} = O(h|\log h|^{1/2}),$$

and upon using $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^\infty(\Omega)} \leq \|\Pi_h\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^\infty(\Omega)} + \|\boldsymbol{u}-\Pi_h\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}$ with (14) and all the previous findings, (23) follows.

Proof. **Estimation for the Gradient of the velocity field** (24)**:** From (22), we have that

$$\|D(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h)\|_{\boldsymbol{L}^\infty(\Omega)} \leq 2|\left\langle D(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h),\boldsymbol{\delta}_{M,2}\right\rangle_\Omega| = 2|\left\langle \Pi_h\boldsymbol{u}-\boldsymbol{u}_h, D\boldsymbol{\delta}_{M,2}\right\rangle_\Omega|,$$

and using the first error equation from (27) tested with $\boldsymbol{v}=\Pi_h\boldsymbol{u}-\boldsymbol{u}_h$, it follows that

$$\begin{aligned}\|D(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h)\|_{\boldsymbol{L}^\infty(\Omega)} \quad&\leq\quad 2\Big|\left\langle\nabla\boldsymbol{G}_2^h,\nabla(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h)\right\rangle_\Omega - \left\langle \textbf{div}\,(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h),\lambda_2^h\right\rangle_\Omega \\ &\qquad + \left\langle\boldsymbol{G}_2^h,\Pi_h\boldsymbol{u}-\boldsymbol{u}_h\right\rangle_\Omega + \left\langle\boldsymbol{b}\cdot\nabla\boldsymbol{G}_2^h,\Pi_h\boldsymbol{u}-\boldsymbol{u}_h\right\rangle_\Omega\Big|.\end{aligned}$$

Using (12), (29) with $q=\lambda_2^h$ and $\boldsymbol{v}=\boldsymbol{u}$ yields $\left\langle\lambda_2^h,\textbf{div}\,(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h)\right\rangle_\Omega = 0$. From the weak form of the error equation given in (29) tested with $\boldsymbol{v}=\boldsymbol{G}_2^h$, it follows that

$$\begin{aligned}\left\langle\nabla(\boldsymbol{u}-\boldsymbol{u}_h),\nabla\boldsymbol{G}_2^h\right\rangle_\Omega + \left\langle\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{G}_2^h\right\rangle_\Omega \quad&=\quad -\left\langle\boldsymbol{b}\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_h),\boldsymbol{G}_2^h\right\rangle_\Omega \\ &\qquad + \left\langle p-p_h,\textbf{div}\,\boldsymbol{G}_2^h\right\rangle_\Omega,\end{aligned}$$

and so

$$\begin{aligned}\|D(\Pi_h\boldsymbol{u}-\boldsymbol{u}_h)\|_{\boldsymbol{L}^\infty(\Omega)} \quad&\leq\quad 2\Big|\left\langle\nabla\boldsymbol{G}_2^h,\nabla(\Pi_h\boldsymbol{u}-\boldsymbol{u})\right\rangle_\Omega + \left\langle\textbf{div}\,\boldsymbol{G}_2^h,p-p_h\right\rangle_\Omega \\ &\qquad + \left\langle\boldsymbol{G}_2^h,\Pi_h\boldsymbol{u}-\boldsymbol{u}\right\rangle_\Omega + \left\langle\boldsymbol{b}\cdot\nabla\boldsymbol{G}_2^h,\Pi_h\boldsymbol{u}-\boldsymbol{u}_h\right\rangle_\Omega \\ &\qquad - \left\langle\boldsymbol{b}\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_h),\boldsymbol{G}_2^h\right\rangle_\Omega\Big|,\end{aligned}$$

hence (24) follows by using Lemma 2 and similar arguments to the ones presented to obtain (23).

Proof. **Estimation for the Pressure field** (25)**:** From (22), it follows that

$$\begin{aligned}\|P_hp-p_h\|_{L^\infty(\Omega)} \quad&\leq\quad 2|\left\langle P_hp-p_h,\delta_{M,3}\right\rangle_\Omega| \\ &=\quad 2\Big(|\left\langle P_hp-p_h,\delta_{M,3}-\phi\right\rangle_\Omega| + |\left\langle P_hp-p_h,\phi\right\rangle_\Omega|\Big).\end{aligned}$$

First notice that using (5) we can bound the second term in the right hand side of the previous inequality as

$$\begin{aligned}|\left\langle P_hp-p_h,\phi\right\rangle_\Omega| \quad&\leq\quad \|P_hp-p_h\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \leq C\|P_hp-p_h\|_{L^2(\Omega)} \\ &\leq\quad C\{\|\nabla(\boldsymbol{u}-\Pi_h\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)} + \|p-P_hp\|_{L^2(\Omega)}\}.\end{aligned}$$

Now, letting $A := \left\langle P_hp-p_h,\delta_{M,3}-\phi\right\rangle_\Omega$ we can bound this term using the Green's function $(\boldsymbol{G}_3,\lambda_3)$ as follows:

$$\begin{aligned}A &= \left\langle P_hp-p_h,\textbf{div}\,\boldsymbol{G}_3^h\right\rangle_\Omega = \left\langle P_hp-p,\textbf{div}\,\boldsymbol{G}_3^h\right\rangle_\Omega + \left\langle p-p_h,\textbf{div}\,\boldsymbol{G}_3^h\right\rangle_\Omega \\ &= \left\langle P_hp-p,\textbf{div}\,(\boldsymbol{G}_3^h-\boldsymbol{G}_3)\right\rangle_\Omega + \left\langle P_hp-p,\textbf{div}\,\boldsymbol{G}_3\right\rangle_\Omega + \left\langle p-p_h,\textbf{div}\,\boldsymbol{G}_3^h\right\rangle_\Omega,\end{aligned}$$

and now using the weak error formulation for $(\boldsymbol{u}, p)$ taking $\boldsymbol{v} = \boldsymbol{G}_3^h$, we have that

$$\left\langle \nabla(\boldsymbol{u}-\boldsymbol{u}_h), \nabla \boldsymbol{G}_3^h \right\rangle_\Omega + \left\langle \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{G}_3^h \right\rangle_\Omega = \left\langle p-p_h, \mathbf{div}\ \boldsymbol{G}_3^h \right\rangle_\Omega , \\ - \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \boldsymbol{G}_3^h \right\rangle_\Omega ,$$

which allows us to rewrite A as

$$\begin{aligned} A &= \left\langle P_h p - p, \mathbf{div}\ (\boldsymbol{G}_3^h - \boldsymbol{G}_3) \right\rangle_\Omega + \left\langle P_h p - p, \mathbf{div}\ \boldsymbol{G}_3 \right\rangle_\Omega + \left\langle \nabla(\boldsymbol{u}-\boldsymbol{u}_h), \nabla \boldsymbol{G}_3^h \right\rangle_\Omega \\ &\quad + \left\langle \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{G}_3^h \right\rangle_\Omega + \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \boldsymbol{G}_3^h \right\rangle_\Omega . \end{aligned}$$

Now, using the weak formulation for $(\boldsymbol{G}_3, \lambda_3)$ with $\boldsymbol{v} = \boldsymbol{u}-\boldsymbol{u}_h$, we obtain

$$\langle \nabla \boldsymbol{G}_3, \nabla(\boldsymbol{u}-\boldsymbol{u}_h)\rangle_\Omega + \langle \boldsymbol{G}_3, \boldsymbol{u}-\boldsymbol{u}_h\rangle_\Omega - \langle \lambda_3, \mathbf{div}\ (\boldsymbol{u}-\boldsymbol{u}_h)\rangle_\Omega + \langle \boldsymbol{b}\cdot\nabla \boldsymbol{G}_3, \boldsymbol{u}-\boldsymbol{u}_h\rangle_\Omega = 0,$$

which yields

$$\begin{aligned} A &= \left\langle P_h p - p, \mathbf{div}\ (\boldsymbol{G}_3^h - \boldsymbol{G}_3) \right\rangle_\Omega + \langle P_h p - p, \mathbf{div}\ \boldsymbol{G}_3 \rangle_\Omega + \langle \lambda_3, \mathbf{div}\ (\boldsymbol{u}-\boldsymbol{u}_h)\rangle_\Omega \\ &\quad + \left\langle \nabla(\boldsymbol{u}-\boldsymbol{u}_h), \nabla(\boldsymbol{G}_3^h - \boldsymbol{G}_3) \right\rangle_\Omega + \left\langle \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{G}_3^h - \boldsymbol{G}_3 \right\rangle_\Omega \\ &\quad + \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \boldsymbol{G}_3^h \right\rangle_\Omega - \langle \boldsymbol{b}\cdot\nabla \boldsymbol{G}_3, \boldsymbol{u}-\boldsymbol{u}_h\rangle_\Omega . \end{aligned}$$

From the second equation of the weak formulation for $(\boldsymbol{u}, p)$ with $q = \lambda_3^h$ and (28), it follows that

$$\begin{aligned} \langle P_h p - p, \mathbf{div}\ \boldsymbol{G}_3\rangle_\Omega &= -\langle \lambda_3, \mathbf{div}\ (\boldsymbol{u}-\boldsymbol{u}_h)\rangle_\Omega + \langle P_h p - p, \delta_{M,3} - \phi\rangle_\Omega \\ &\quad + \left\langle \lambda_3 - \lambda_3^h, \mathbf{div}\ (\boldsymbol{u}-\boldsymbol{u}_h)\right\rangle_\Omega , \end{aligned}$$

and using the weak form related to the error equation for $(\boldsymbol{G}_3, \lambda_3)$ tested with $\boldsymbol{v} = \boldsymbol{u}_h - \Pi_h \boldsymbol{u}$ gives

$$\begin{aligned} \left\langle \nabla(\boldsymbol{G}_3 - \boldsymbol{G}_3^h), \nabla(\boldsymbol{u}_h - \Pi_h \boldsymbol{u})\right\rangle_\Omega &= -\left\langle \boldsymbol{G}_3 - \boldsymbol{G}_3^h, \boldsymbol{u}_h - \Pi_h \boldsymbol{u}\right\rangle_\Omega \\ &\quad + \left\langle \lambda_3 - \lambda_3^h, \mathbf{div}\ (\boldsymbol{u}_h - \Pi_h \boldsymbol{u})\right\rangle_\Omega \\ &\quad - \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{G}_3 - \boldsymbol{G}_3^h), \boldsymbol{u}_h - \Pi_h \boldsymbol{u}\right\rangle_\Omega . \end{aligned}$$

Noticing that

$$\begin{aligned} &-\langle \boldsymbol{b}\cdot\nabla \boldsymbol{G}_3, \boldsymbol{u}-\boldsymbol{u}_h\rangle_\Omega - \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{G}_3 - \boldsymbol{G}_3^h), \boldsymbol{u}_h - \Pi_h \boldsymbol{u}\right\rangle_\Omega \\ &= \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{u} - \Pi_h \boldsymbol{u}), \boldsymbol{G}_3 - \boldsymbol{G}_3^h\right\rangle_\Omega + 2\left\langle \boldsymbol{b}\cdot\nabla(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h), \boldsymbol{G}_3^h\right\rangle_\Omega \end{aligned}$$

by a Hölder inequality, allows us to conclude that

$$\begin{aligned} A &\le C\Big(\{ \|\nabla(\boldsymbol{G}_3 - \boldsymbol{G}_3^h)\|_{\boldsymbol{L}^1(\Omega)} + \|\boldsymbol{G}_3 - \boldsymbol{G}_3^h\|_{\boldsymbol{L}^1(\Omega)} + \|\delta_{M,3} - \phi\|_{L^1(\Omega)} + h\|\nabla\lambda_3\|_{L^1(\Omega)} \} \\ &\quad \times \{ \|\nabla(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{\boldsymbol{L}^\infty(\Omega)} + \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} + \|p - P_h p\|_{L^\infty(\Omega)} \} \\ &\quad + \|\Pi_h \boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{G}_3^h\|_{\boldsymbol{L}^2(\Omega)} \Big). \end{aligned}$$

Again using a Hölder inequality together with Lemma 3 and Lemma 2, we conclude that

$$\|\nabla(\boldsymbol{G}_3^h - \boldsymbol{G}_3)\|_{\boldsymbol{L}^1(\Omega)} \le C|\log h|^{1/2}, \quad \|\boldsymbol{G}_3^h - \boldsymbol{G}_3\|_{\boldsymbol{L}^1(\Omega)} \le |\log h|^{1/2}, \\ h\|\nabla\lambda_3\|_{L^1(\Omega)} \le C|\log h|^{1/2},$$

and since $\|\Pi_h \boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)} \|\nabla \boldsymbol{G}_3^h\|_{\boldsymbol{L}^2(\Omega)} = O(1)$, yields

$$\begin{aligned} |\langle P_h p - p_h, \delta_{M,3} - \phi\rangle_\Omega| &\le C|\log h| \Big\{ \|\nabla(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{\boldsymbol{L}^\infty(\Omega)} \\ &\quad + \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} + \|p - P_h p\|_{L^\infty(\Omega)} \Big\}, \end{aligned}$$

which combined with (14) and a triangle inequality allows us to conclude (25).

3. Proof of auxiliary results

In this section we will focus on proving all the auxiliary results used to obtain Theorem 1, that is proving Lemma 1 to 3 and Theorem 2 .

Proof. **Lemma 1**: Letting $\mu_j = \boldsymbol{x}^j - \boldsymbol{x}_i^j$ be the difference of each component of the vector $\boldsymbol{x} - \boldsymbol{x}_i$, it is possible to establish that

$$\begin{array}{rcl} \nabla(\mu_j \lambda_1) &=& \mu_j \nabla \lambda_1 + \lambda_1 \nabla \mu_j, \\ \nabla(\mu_j \boldsymbol{G}_1) &=& \mu_j \nabla \boldsymbol{G}_1 + \boldsymbol{G}_1 \nabla \mu_j, \\ \Delta(\mu_j \boldsymbol{G}_1) &=& \mu_j \Delta \boldsymbol{G}_1 + \boldsymbol{G}_1 \Delta \mu_j + 2\nabla \mu_j \cdot \nabla \boldsymbol{G}_1, \end{array} \tag{35}$$

hence the following upper bound follows:

$$\begin{aligned} &\|\mu_j \nabla \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\mu_j \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\mu_j D^2 \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\mu_j \nabla \lambda_1\|_{\boldsymbol{L}^2(\Omega)} \\ &\quad \leq C\Big(\|D^2(\mu_j \boldsymbol{G}_1)\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\mu_j \boldsymbol{G}_1)\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\mu_j \lambda_1)\|_{\boldsymbol{L}^2(\Omega)} + \|\lambda_1\|_{L^2(\Omega)} \\ &\qquad + \|\nabla \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\mu_j \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} \Big). \end{aligned}$$

Now, multiplying (26) by μ_j and using (35) yields the following auxiliary problem

$$\left\{ \begin{array}{rcll} -\Delta(\mu_j \boldsymbol{G}_1) + \boldsymbol{b} \cdot \nabla(\boldsymbol{G}_1 \mu_j) + \nabla(\mu_j \lambda_1) + \boldsymbol{G}_1 \mu_j &=& \mu_j \boldsymbol{\delta}_{M,1} - 2\nabla \mu_j \cdot \nabla \boldsymbol{G}_1 & \\ && + \boldsymbol{b} \cdot (\boldsymbol{G}_1 \nabla \mu_j) + \lambda_1 \nabla \mu_j & \text{in } \Omega, \\ \textbf{div}\ (\boldsymbol{G}_1 \mu_j) &=& \nabla \mu_j \cdot \boldsymbol{G}_1 & \text{in } \Omega, \end{array} \right.$$

and since $\nabla \mu_j \cdot \boldsymbol{G}_1 \in H_0^1(\Omega)$, by using (7) we can conclude that

$$\begin{aligned} &\|\mu_j \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|D^2(\mu_j \boldsymbol{G}_1)\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\mu_j \boldsymbol{G}_1)\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\mu_j \lambda_1)\|_{\boldsymbol{L}^2(\Omega)} \\ &\quad \leq C\{\|\mu_j \boldsymbol{\delta}_{M,1}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\lambda_1\|_{L^2(\Omega)}\}. \end{aligned}$$

Noticing that $\|\mu_j \boldsymbol{\delta}_{M,1}\|_{\boldsymbol{L}^2(\Omega)} = O(1)$ (upon using (19)-(21)), from the inf-sup condition, denoting by $\boldsymbol{S}$ the unit ball in $\boldsymbol{H}_0^1(\Omega)$ and taking I_h to be the local average interpolant given in page 109 from [15], it follows that

$$\begin{aligned} \|\boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\lambda_1\|_{L^2(\Omega)} &\leq C\|\boldsymbol{\delta}_{M,1}\|_{\boldsymbol{H}^{-1}(\Omega)} = C \sup_{\boldsymbol{v} \in \boldsymbol{S}} \langle \boldsymbol{\delta}_{M,1}, \boldsymbol{v} \rangle_\Omega \\ &\leq \sup_{\boldsymbol{v} \in \boldsymbol{S}} \langle \boldsymbol{\delta}_{M,1}, \boldsymbol{v} - I_h \boldsymbol{v} \rangle_\Omega \leq C(1 + |\log h|^{1/2}), \end{aligned}$$

hence

$$\|\mu_j \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\mu_j D^2 \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\mu_j \nabla \boldsymbol{G}_1\|_{\boldsymbol{L}^2(\Omega)} + \|\mu_j \nabla \lambda_j\|_{\boldsymbol{L}^2(\Omega)} \quad \leq \quad C|\log h|^{1/2},$$

and the result follows upon collecting all of the previous results and the following expressions

$$\begin{array}{rcl} \|\boldsymbol{G}_1\|^2_{\sigma_1^2} &=& \theta^2 \|\boldsymbol{G}_1\|^2_{\boldsymbol{L}^2(\Omega)} + \displaystyle\sum_{j=1}^2 \|\mu_j \boldsymbol{G}_1\|^2_{\boldsymbol{L}^2(\Omega)}, \\ \|\nabla \boldsymbol{G}_1\|^2_{\sigma_1^2} &=& \theta^2 \|\nabla \boldsymbol{G}_1\|^2_{\boldsymbol{L}^2(\Omega)} + \displaystyle\sum_{j=1}^2 \|\mu_j \nabla \boldsymbol{G}_1\|^2_{\boldsymbol{L}^2(\Omega)}, \\ \|D^2 \boldsymbol{G}_1\|^2_{\sigma_1^2} &=& \theta^2 \|D^2 \boldsymbol{G}_1\|^2_{\boldsymbol{L}^2(\Omega)} + \displaystyle\sum_{j=1}^2 \|\mu_j D^2 \boldsymbol{G}_1\|^2_{\boldsymbol{L}^2(\Omega)}, \\ \|\nabla \lambda_1\|^2_{\sigma_1^2} &=& \theta^2 \|\nabla \lambda_1\|^2_{\boldsymbol{L}^2(\Omega)} + \displaystyle\sum_{j=1}^2 \|\mu_j \nabla \lambda_1\|^2_{\boldsymbol{L}^2(\Omega)}, \end{array} \tag{36}$$

where $\theta = Kh$ and also using (7) to estimate the terms $\theta^2 \|\boldsymbol{G}_1\|^2_{\boldsymbol{L}^2(\Omega)}$, $\theta^2 \|\nabla \boldsymbol{G}_1\|^2_{\boldsymbol{L}^2(\Omega)}$, $\theta^2 \|D^2 \boldsymbol{G}_1\|^2_{\boldsymbol{L}^2(\Omega)}$ and $\theta^2 \|\nabla \lambda_1\|^2_{\boldsymbol{L}^2(\Omega)}$.

Proof. **Lemma 2**: Following closely the proof of the previous lemma, i.e., replacing $\boldsymbol{G}_1$ by $\boldsymbol{G}_2$, λ_1 by λ_2 and $\boldsymbol{\delta}_{M,1}$ by $D\boldsymbol{\delta}_{M,2}$ it is possible to conclude that

$$\begin{aligned}&\|\mu_j\boldsymbol{G}_2\|_{\boldsymbol{L}^2(\Omega)}+\|D^2(\mu_j\boldsymbol{G}_2)\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\mu_j\boldsymbol{G}_2)\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\mu_j\lambda_2)\|_{\boldsymbol{L}^2(\Omega)}\\&\quad\le C\{\|\mu_jD\boldsymbol{\delta}_{M,2}\|_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{G}_2\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{G}_2\|_{\boldsymbol{L}^2(\Omega)}+\|\lambda_2\|_{L^2(\Omega)}\}.\end{aligned}$$

Againg using (19)-(21) it follows that $\|\mu_jD\boldsymbol{\delta}_{M,2}\|_{\boldsymbol{L}^2(\Omega)}=O(1)$ and $\|\boldsymbol{\delta}_{M,2}\|^2_{\boldsymbol{L}^2(\Omega)}\le Ch^{-2}$ (upon using (21)), which yields the following estimate

$$\|\boldsymbol{G}_2\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{G}_2\|_{\boldsymbol{L}^2(\Omega)}+\|\lambda_2\|_{L^2(\Omega)}\le C\|D\boldsymbol{\delta}_{M,2}\|_{\boldsymbol{H}^{-1}(\Omega)}\le C\|\boldsymbol{\delta}_{M,2}\|_{\boldsymbol{L}^2(\Omega)}\le Ch^{-1},$$

which allows us to conclude that

$$\|\mu_j\boldsymbol{G}_2\|_{\boldsymbol{L}^2(\Omega)}+\|D^2(\mu_j\boldsymbol{G}_2)\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\mu_j\boldsymbol{G}_2)\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\mu_j\lambda_2)\|_{\boldsymbol{L}^2(\Omega)}\le Ch^{-1},$$

and the result follows using (36) and (7).

Proof. **Lemma 3**: As in the proof of the previous two lemmas, it follows that

$$\begin{aligned}&\|\mu_j\boldsymbol{G}_3\|_{\boldsymbol{L}^2(\Omega)}+\|D^2(\mu_j\boldsymbol{G}_3)\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\mu_j\boldsymbol{G}_3)\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\mu_j\lambda_3)\|_{\boldsymbol{L}^2(\Omega)}\\&\qquad\le C\left(\|\boldsymbol{G}_3\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{G}_3\|_{\boldsymbol{L}^2(\Omega)}+\|\lambda_3\|_{L^2(\Omega)}\right),\end{aligned}$$

and using (19)-(21) we can conclude that

$$\|\boldsymbol{G}_3\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{G}_3\|_{\boldsymbol{L}^2(\Omega)}+\|\lambda_3\|_{L^2(\Omega)}\le C\|\delta_{M,3}-\phi\|_{L^2(\Omega)}\le Ch^{-1},$$

from which the result follows using the same arguments as before.

Proof. **Theorem 2**: To begin the analysis let us define

$$\boldsymbol{\psi}\;=\;\sigma^2(\boldsymbol{G}-\boldsymbol{G}^h).\tag{37}$$

Then

$$\begin{aligned}\|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|^2_{\sigma^2}&=\left\langle\nabla(\boldsymbol{G}-\boldsymbol{G}^h),\sigma^2\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\right\rangle_\Omega\\&=\left\langle\nabla(\boldsymbol{G}-\boldsymbol{G}^h),\nabla\boldsymbol{\psi}\right\rangle_\Omega+\frac12\left\langle\boldsymbol{G}-\boldsymbol{G}^h,\Delta\sigma^2\cdot(\boldsymbol{G}-\boldsymbol{G}^h)\right\rangle_\Omega.\end{aligned}$$

Using the estimate

$$\begin{aligned}\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\boldsymbol{L}^2(\Omega)}&\le Ch\left(\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\|\boldsymbol{G}-\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}+\inf_{q\in Q_h}\|\lambda-q\|_{L^2(\Omega)}\right)\\&\le Ch\left(\|\boldsymbol{G}-\Pi_h\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\lambda-P_h\lambda\|_{L^2(\Omega)}\right)\\&\le Ch^2\left(\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{\boldsymbol{L}^2(\Omega)}\right),\end{aligned}\tag{38}$$

which can be obtained upon using (6), (5), (13) and (17), we have that

$$\|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|^2_{\sigma^2}\le\left\langle\nabla(\boldsymbol{G}-\boldsymbol{G}^h),\nabla\boldsymbol{\psi}\right\rangle_\Omega+Ch^4\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{L^2(\Omega)}\}^2,\tag{39}$$

where $A:=\left\langle\nabla(\boldsymbol{G}-\boldsymbol{G}^h),\nabla\boldsymbol{\psi}\right\rangle_\Omega$ and $B:=Ch^4\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{L^2(\Omega)}\}^2$. Now, to bound A we use the weak form for the error equation for $(\boldsymbol{G},\lambda)$ given in (30) tested with $\boldsymbol{v}=\Pi_h\boldsymbol{\psi}$ and $q=\lambda^h-P_h\lambda$, and also using (12) (tested with $\boldsymbol{v}=\boldsymbol{\psi}$) it is possible to conclude that

$$\begin{aligned}A\;=\;&\left\langle\nabla(\boldsymbol{G}-\boldsymbol{G}^h),\nabla\boldsymbol{\psi}\right\rangle_\Omega+\langle\lambda-\lambda_h,\mathbf{div}\;\Pi_h\boldsymbol{\psi}\rangle_\Omega-\left\langle\nabla(\boldsymbol{G}-\boldsymbol{G}^h),\nabla\Pi_h\boldsymbol{\psi}\right\rangle_\Omega\\&-\left\langle\boldsymbol{G}-\boldsymbol{G}^h,\Pi_h\boldsymbol{\psi}\right\rangle_\Omega-\langle\lambda_h-P_h\lambda,\mathbf{div}\;(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})\rangle_\Omega\\&-\left\langle\boldsymbol{b}\cdot\nabla(\boldsymbol{G}-\boldsymbol{G}^h),\Pi_h\boldsymbol{\psi}\right\rangle_\Omega.\end{aligned}$$

Using integration by parts and (39), (37) and a Hölder inequality, there exists a positive number ϵ which is independent of h such that

$$|\langle \boldsymbol{b} \cdot \nabla(\boldsymbol{G} - \boldsymbol{G}^h), \boldsymbol{\psi}\rangle_\Omega| = |\langle \sigma \boldsymbol{b} \cdot \nabla \boldsymbol{\psi}, \sigma^{-1}(\boldsymbol{G} - \boldsymbol{G}^h)\rangle_\Omega| \leq C\|\nabla \boldsymbol{\psi}\|^2_{\sigma^{-2}} + \epsilon \|\boldsymbol{G} - \boldsymbol{G}^h\|^2_{\sigma^2},$$

which follows using the fact that $\boldsymbol{\psi} \in H^1_0(\Omega)$ and integration by parts. Hence, we can finally conclude that there exist a constant $C > 0$ such that

$$\begin{aligned}
&\|\boldsymbol{G} - \boldsymbol{G}^h\|^2_{\sigma^2} + \|\nabla(\boldsymbol{G} - \boldsymbol{G}^h)\|^2_{\sigma^2} \\
&\leq Ch^4\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\lambda\|_{L^2(\Omega)}\}^2 + \epsilon\|\nabla(\boldsymbol{G} - \boldsymbol{G}^h)\|^2_{\sigma^2} + C\|\nabla(\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi})\|^2_{\sigma^{-2}} \\
&\qquad + \|\lambda - P_h\lambda\|^2_{\sigma^2} + C\|\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\nabla\boldsymbol{\psi}\|^2_{\sigma^{-2}} + \epsilon\|\boldsymbol{G} - \boldsymbol{G}^h\|^2_{\sigma^2} \\
&\qquad\quad + |\langle \mathbf{div}\ \ \boldsymbol{\psi}, \lambda - \lambda^h\rangle_\Omega| + C\|\nabla(\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi})\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{G} - \boldsymbol{G}^h\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

The result then follows using Lemma 4-10, presented below, which allows us to bound $|\langle \mathbf{div}\ \ \boldsymbol{\psi}, \lambda - \lambda^h\rangle_\Omega|$, $\|\nabla\boldsymbol{\psi}\|^2_{\sigma^{-2}}$, $\|\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi}\|^2_{L^2(\Omega)}$, $\|\nabla(\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi})\|^2_{L^2(\Omega)}$, $\|\boldsymbol{G} - \boldsymbol{G}^h\|^2_{\boldsymbol{L}^2(\Omega)}$, $\|\lambda - P_h\lambda\|^2_{\sigma^2}$ and $\|\nabla(\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi})\|^2_{\sigma^{-2}}$.

Lemma 4. *For K defined in* (8) *and $\boldsymbol{\psi}$ given by* (37)*, there exist constants $C_K > 0$ and $C > 0$ independent of K such that*

$$\begin{aligned}
\|\nabla(\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi})\|_{\sigma^{-2}} \quad \leq \quad & C_K h^2\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\lambda\|_{L^2(\Omega)}\} \\
& + C_K h\{\|D^2\boldsymbol{G}\|_{\sigma^2} + \|\nabla\lambda\|_{\sigma^2}\} + \frac{C}{K}\|\nabla(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2}.
\end{aligned}$$

Proof. First, we rewrite $\boldsymbol{\psi}$ as

$$\boldsymbol{\psi} = \sigma^2(\boldsymbol{G} - \boldsymbol{G}^h) = \sigma^2(\boldsymbol{G} - \Pi_h\boldsymbol{G}) + \sigma^2(\Pi_h\boldsymbol{G} - \boldsymbol{G}^h) := \boldsymbol{\psi}_1 + \boldsymbol{\psi}_2 \tag{40}$$

then

$$\|\nabla(\boldsymbol{\psi} - \Pi_h\boldsymbol{\psi})\|_{\sigma^{-2}} \leq \|\nabla(\boldsymbol{\psi}_1 - \Pi_h\boldsymbol{\psi}_1)\|_{\sigma^{-2}} + \|\nabla(\boldsymbol{\psi}_2 - \Pi_h\boldsymbol{\psi}_2)\|_{\sigma^{-2}}.$$

From (15), we obtain $\|\nabla(\boldsymbol{\psi}_1 - \Pi_h\boldsymbol{\psi}_1)\|_{\sigma^{-2}} \leq C\|\nabla\boldsymbol{\psi}_1\|_{\sigma^{-2}}$, and using (13) and (9) we have that

$$\begin{aligned}
\|\nabla\boldsymbol{\psi}_1\|_{\sigma^{-2}} &\leq C\left\{\|\boldsymbol{G} - \Pi_h\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla(\boldsymbol{G} - \Pi_h\boldsymbol{G})\|_{\sigma^2}\right\} \\
&\leq Ch^2\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)} + Ch\|D^2\boldsymbol{G}\|_{\sigma^2}.
\end{aligned}$$

For the second term $\|\nabla(\boldsymbol{\psi}_2 - \Pi_h\boldsymbol{\psi}_2)\|_{\sigma^{-2}}$ using a triangle inequality and (13) with (8), yields

$$\begin{aligned}
\|\nabla(\boldsymbol{\psi}_2 - \Pi_h\boldsymbol{\psi}_2)\|_{\sigma^{-2}} &\leq Ch\|\Pi_h\boldsymbol{G} - \boldsymbol{G}^h\|_{\sigma^{-2}} + Ch\|\nabla(\Pi_h\boldsymbol{G} - \boldsymbol{G}^h)\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq \frac{C}{K}\|\Pi_h\boldsymbol{G} - \boldsymbol{G}^h\|_{\boldsymbol{L}^2(\Omega)} + \frac{C}{K}\|\nabla(\Pi_h\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2} \\
&\leq \frac{C}{K}\left(\|\Pi_h\boldsymbol{G} - \boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{G} - \boldsymbol{G}^h\|_{\boldsymbol{L}^2(\Omega)}\right) \\
&\quad + \frac{C}{K}\|\nabla(\Pi_h\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2},
\end{aligned} \tag{41}$$

and inserting (38) into (41), it follows that

$$\begin{aligned}
\|\nabla(\boldsymbol{\psi}_2 - \Pi_h\boldsymbol{\psi}_2)\|_{\sigma^{-2}} \quad \leq \quad & \frac{C}{K}h^2\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\lambda\|_{L^2(\Omega)}\} \\
& + \frac{C}{K}h\|D^2\boldsymbol{G}\|_{\sigma^2} + \frac{C}{K}\|\nabla(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2},
\end{aligned}$$

from which the result follows upon combining all of the previous bounds.

Lemma 5. *Let* $\boldsymbol{\psi}$ *given by* (37). *Then, there exists a constant* $C > 0$ *such that*

$$\begin{aligned} |\left\langle \mathbf{div}\ \psi, \lambda - \lambda^h \right\rangle_\Omega| \quad &\leq \quad Ch^2\Big(\{\|D^2\boldsymbol{G}\|_{\sigma^2} + \|\nabla\lambda\|_{\sigma^2}\}^2 \\ &\quad + h^2\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\lambda\|_{L^2(\Omega)}\}^2\Big) + \epsilon\|\nabla(\boldsymbol{G} - \boldsymbol{G}^h)\|^2_{\sigma^2}. \end{aligned}$$

PROOF. Since $\mathbf{div}\ \psi = \nabla\sigma^2 \cdot (\boldsymbol{G} - \boldsymbol{G}^h) + \sigma^2\mathbf{div}\ (\boldsymbol{G} - \boldsymbol{G}^h)$, it follows that

$$\begin{aligned} \left\langle \mathbf{div}\ \psi, \lambda - \lambda^h \right\rangle_\Omega &= \left\langle \nabla\sigma^2 \cdot (\boldsymbol{G} - \boldsymbol{G}^h), \lambda - \lambda^h \right\rangle_\Omega + \left\langle \mathbf{div}\ (\boldsymbol{G} - \boldsymbol{G}^h), \sigma^2(\lambda - \lambda^h) \right\rangle_\Omega \\ &= \gamma\left\langle \phi, \lambda - \lambda^h \right\rangle_\Omega + \left\langle g, \lambda - \lambda^h \right\rangle_\Omega + \left\langle \mathbf{div}\ (\boldsymbol{G} - \boldsymbol{G}^h), \sigma^2(\lambda - \lambda^h) \right\rangle_\Omega, \end{aligned}$$

where we take

$$\gamma := \int_\Omega \nabla\sigma^2 \cdot (\boldsymbol{G} - \boldsymbol{G}^h) d\boldsymbol{x}, \tag{42}$$

$$g := \nabla\sigma^2 \cdot (\boldsymbol{G} - \boldsymbol{G}^h) - \gamma\phi, \tag{43}$$

with $\phi \in C_0^\infty(\Omega)$. Then, the result follows upon using Lemma 6, 7 and 8, presented below.

Lemma 6. *For* γ *given by* (42), *there exists a constant* $C > 0$ *such that*

$$|\gamma| \leq Ch\|\nabla(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2}.$$

PROOF. Using integration by parts and (17), we have that

$$|\gamma| = \left|\left\langle \mathbf{div}\ (\boldsymbol{G} - \boldsymbol{G}^h), \sigma^2 \right\rangle_\Omega\right| = \left|\left\langle \mathbf{div}\ (\boldsymbol{G} - \boldsymbol{G}^h), \sigma^2 - P_h\sigma^2 \right\rangle_\Omega\right| \leq Ch\|\nabla(\boldsymbol{G} - \boldsymbol{G}^h)\|_{\sigma^2},$$

upon using the error equation in (30) for $\boldsymbol{G}$.

Lemma 7. *For* g *given by* (43), *there exists a constant* $C > 0$ *such that*

$$\left|\left\langle g, \lambda - \lambda^h \right\rangle_\Omega\right| \leq Ch^4\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\lambda\|_{L^2(\Omega)}\}^2 + \epsilon\|\nabla(\boldsymbol{G} - \boldsymbol{G}^h)\|^2_{\sigma^2}.$$

PROOF. We use a duality argument. Let us consider the following auxiliary problem

$$\left\{ \begin{array}{c} -\Delta\boldsymbol{w} + \boldsymbol{b}\cdot\nabla\boldsymbol{w} + \nabla\eta + \boldsymbol{w} = \boldsymbol{0} \quad \text{and} \quad \mathbf{div}\ \ \boldsymbol{w} = g \quad \text{in } \Omega, \\ \boldsymbol{w} = \boldsymbol{0} \quad \text{on } \partial\Omega. \end{array} \right. \tag{44}$$

If $g \in H_0^1(\Omega)$ and $\langle g, 1\rangle_\Omega = 0$ for a convex polygon Ω, then it follows that $\boldsymbol{w} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ and $\eta \in H^1(\Omega) \cap L_0^2(\Omega)$, hence

$$\|\boldsymbol{w}\|_{\boldsymbol{H}^2(\Omega)} + \|\nabla\eta\|_{L^2(\Omega)} \quad \leq \quad C\|\nabla g\|_{L^2(\Omega)}. \tag{45}$$

Now, using the weak error equation for $(\boldsymbol{G}, \lambda)$ tested with $\boldsymbol{v} = \boldsymbol{w}^h$, it follows that

$$\begin{aligned} \left\langle \nabla(\boldsymbol{G} - \boldsymbol{G}^h), \nabla\boldsymbol{w}_h \right\rangle_\Omega + \left\langle \boldsymbol{G} - \boldsymbol{G}^h, \boldsymbol{w}_h \right\rangle_\Omega &= \left\langle \lambda - \lambda^h, \mathbf{div}\ \ \boldsymbol{w}_h \right\rangle_\Omega \\ &- \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{G} - \boldsymbol{G}^h), \boldsymbol{w}_h \right\rangle_\Omega, \end{aligned} \tag{46}$$

and from the second weak error equation for $(\boldsymbol{G}, \lambda)$ tested with $q = \eta_h$, we have that

$$\left\langle \eta_h, \mathbf{div}\ (\boldsymbol{G} - \boldsymbol{G}^h) \right\rangle_\Omega = 0. \tag{47}$$

Using the weak formulation of the auxiliary problem (44) tested with $\boldsymbol{v} = \boldsymbol{G} - \boldsymbol{G}^h$ we also have that

$$\begin{aligned} \left\langle \nabla\boldsymbol{w}, \nabla(\boldsymbol{G} - \boldsymbol{G}^h) \right\rangle_\Omega + \left\langle \boldsymbol{b}\cdot\nabla\boldsymbol{w}, \boldsymbol{G} - \boldsymbol{G}^h \right\rangle_\Omega &= \left\langle \eta, \mathbf{div}\ (\boldsymbol{G} - \boldsymbol{G}^h) \right\rangle_\Omega \\ &- \left\langle \boldsymbol{w}, \boldsymbol{G} - \boldsymbol{G}^h \right\rangle_\Omega, \end{aligned} \tag{48}$$

and now subtracting (46) from (48) and (47), we have

$$\begin{aligned}
|\left\langle g,\lambda-\lambda^h\right\rangle_\Omega| &= |\left\langle \mathbf{div}\,\boldsymbol{w},\lambda-\lambda^h\right\rangle_\Omega| \\
&= \Big|\left\langle \mathbf{div}\,(\boldsymbol{w}-\boldsymbol{w}_h),\lambda-\lambda^h\right\rangle_\Omega - \left\langle \nabla(\boldsymbol{G}-\boldsymbol{G}^h),\nabla(\boldsymbol{w}-\boldsymbol{w}_h)\right\rangle_\Omega \\
&\quad -\left\langle \boldsymbol{G}-\boldsymbol{G}^h,\boldsymbol{w}-\boldsymbol{w}_h\right\rangle_\Omega + \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{G}-\boldsymbol{G}^h),\boldsymbol{w}_h\right\rangle_\Omega \\
&\quad -\left\langle \boldsymbol{b}\cdot\nabla\boldsymbol{w},\boldsymbol{G}-\boldsymbol{G}^h\right\rangle_\Omega - \left\langle \mathbf{div}\,(\boldsymbol{G}-\boldsymbol{G}^h),\eta_h-\eta\right\rangle_\Omega\Big|,
\end{aligned}$$

which allows us to conclude that

$$\begin{aligned}
|\left\langle g,\lambda-\lambda^h\right\rangle_\Omega| &= \Big|\left\langle \mathbf{div}\,(\boldsymbol{w}-\boldsymbol{w}_h),\lambda-\lambda^h\right\rangle_\Omega - \left\langle \nabla(\boldsymbol{G}-\boldsymbol{G}^h),\nabla(\boldsymbol{w}-\boldsymbol{w}_h)\right\rangle_\Omega \\
&\quad -\left\langle \boldsymbol{G}-\boldsymbol{G}^h,\boldsymbol{w}-\boldsymbol{w}_h\right\rangle_\Omega - \left\langle \boldsymbol{b}\cdot\nabla(\boldsymbol{G}-\boldsymbol{G}^h),\boldsymbol{w}-\boldsymbol{w}_h\right\rangle_\Omega \\
&\quad -2\left\langle \boldsymbol{b}\cdot\nabla\boldsymbol{w},\boldsymbol{G}-\boldsymbol{G}^h\right\rangle_\Omega - \left\langle \mathbf{div}\,(\boldsymbol{G}-\boldsymbol{G}^h),\eta_h-\eta\right\rangle_\Omega\Big|.
\end{aligned}$$

Using a Hölder inequality, (5), (6), (13), (17) and (45) it follows that

$$\begin{aligned}
|\left\langle g,\lambda-\lambda^h\right\rangle_\Omega| &\leq \|\nabla(\boldsymbol{w}-\boldsymbol{w}_h)\|_{\boldsymbol{L}^2(\Omega)}\|\lambda-\lambda_h\|_{L^2(\Omega)} \\
&\quad + \|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\boldsymbol{L}^2(\Omega)}\|\nabla(\boldsymbol{w}-\boldsymbol{w}_h)\|_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \|\boldsymbol{G}-\boldsymbol{G}^h\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}-\boldsymbol{w}_h\|_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{w}-\boldsymbol{w}_h\|_{\boldsymbol{L}^2(\Omega)} \\
&\quad + \|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\boldsymbol{L}^2(\Omega)}\|\eta-\eta_h\|_{L^2(\Omega)} \\
&\quad + 2\left\langle \boldsymbol{b}\cdot\nabla\boldsymbol{w},\boldsymbol{G}-\boldsymbol{G}^h\right\rangle_\Omega \\
&\leq C\Big(h\left\{\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\eta\|_{L^2(\Omega)}\right\}h\left\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{L^2(\Omega)}\right\} \\
&\quad + \|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\boldsymbol{L}^2(\Omega)}\Big) \\
&\leq Ch^2\|\nabla g\|_{L^2(\Omega)}\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{L^2(\Omega)}\},
\end{aligned}$$

and using the definition of g given in (43) together with (9), it follows that

$$\|\nabla g\|_{L^2(\Omega)} \leq Ch^2\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{L^2(\Omega)}\} + C\|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2} + C|\gamma|.$$

By combining the previous two bounds with Lemma 6 the result follows.

Lemma 8. *There exists a constant $C>0$ such that*

$$\begin{aligned}
\left|\left\langle \mathbf{div}\,(\boldsymbol{G}-\boldsymbol{G}^h),\sigma^2(\lambda-\lambda^h)\right\rangle_\Omega\right| &\leq Ch^2\|\nabla\lambda\|^2_{\sigma^2} + \epsilon\|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|^2_{\sigma^2} \\
&\quad + Ch^4\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|^2_{L^2(\Omega)}\}.
\end{aligned}$$

Proof. Taking $\xi=\sigma^2(P_h\lambda-\lambda^h)$, and using the weak error form for $(\boldsymbol{G},\lambda)$ tested with $q=\xi-P_h\xi$ and (17), it follows that

$$\begin{aligned}
\left|\left\langle \mathbf{div}\,(\boldsymbol{G}-\boldsymbol{G}^h),\sigma^2(\lambda-\lambda^h)\right\rangle_\Omega\right| &\leq \left|\left\langle \mathbf{div}\,(\boldsymbol{G}-\boldsymbol{G}^h),\sigma^2(\lambda-P_h\lambda)+\xi-P_h\xi\right\rangle_\Omega\right| \\
&\leq Ch\|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}\|\nabla\lambda\|_{\sigma^2} \\
&\quad + \|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}\|\xi-P_h\xi\|_{\sigma^{-2}},
\end{aligned}$$

and now using (5), (18), (17) with the fact that $\|P_h\lambda-\lambda^h\|_{L^2(\Omega)} \leq \|P_h\lambda-\lambda\|_{L^2(\Omega)} + \|\lambda-\lambda^h\|_{L^2(\Omega)}$, yields

$$\|\xi-P_h\xi\|_{\sigma^{-2}} \leq Ch\|P_h\lambda-\lambda^h\|_{L^2(\Omega)} \leq Ch^2\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{\boldsymbol{L}^2(\Omega)}\},$$

from which the result follows.

Lemma 9. *Let $\boldsymbol{\psi}$ be given by* (37)*. Then, there exists a constant $C>0$ such that*

$$\begin{aligned}\|\nabla\boldsymbol{\psi}\|_{\sigma^{-2}} &\leq Ch^2\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{L^2(\Omega)}\}\\ &\quad +Ch\{\|D^2\boldsymbol{G}\|_{\sigma^2}+\|\nabla\lambda\|_{\sigma^2}\}+C\|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}.\end{aligned}$$

PROOF. First, noticing that by using (40) it follows that $\|\nabla\boldsymbol{\psi}\|_{\sigma^{-2}}\leq\|\nabla\boldsymbol{\psi}_1\|_{\sigma^{-2}}+\|\nabla\boldsymbol{\psi}_2\|_{\sigma^{-2}}$, which combined with (13) and (9) yields

$$\begin{aligned}\|\nabla\boldsymbol{\psi}_1\|_{\sigma^{-2}} &\leq C\left\{\|\boldsymbol{G}-\Pi_h\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla(\boldsymbol{G}-\Pi_h\boldsymbol{G})\|_{\sigma^2}\right\}\\ &\leq Ch^2\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+Ch\|D^2\boldsymbol{G}\|_{\sigma^2}.\end{aligned}$$

Now, for the second term $\|\nabla\boldsymbol{\psi}_2\|_{\sigma^{-2}}$ we have that

$$\begin{aligned}\|\nabla\boldsymbol{\psi}_2\|_{\sigma^{-2}} &\leq C\|\Pi_h\boldsymbol{G}-\boldsymbol{G}^h\|_{\boldsymbol{L}^2(\Omega)}+C\|\nabla(\Pi_h\boldsymbol{G}-\boldsymbol{G})\|_{\sigma^2}\\ &\leq C\left(\|\Pi_h\boldsymbol{G}-\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\boldsymbol{G}-\boldsymbol{G}^h\|_{\boldsymbol{L}^2(\Omega)}\right)+C\|\nabla(\Pi_h\boldsymbol{G}-\boldsymbol{G})\|_{\sigma^2}\\ &\leq Ch^2\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{L^2(\Omega)}\}+Ch\|D^2\boldsymbol{G}\|_{\sigma^2}+C\|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2},\end{aligned}$$

upon using similar arguments as before. The result then follows.

Lemma 10. *Let K and $\boldsymbol{\psi}$ be given by* (8) *and* (37)*, respectively. Then, there exists a constant $C_K>0$ and a constant $C>0$ independent of K such that*

$$\begin{aligned}\|\nabla(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})\|_{\boldsymbol{L}^2(\Omega)} &\leq C_Kh^2\Big(h\{\|D^2\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\lambda\|_{L^2(\Omega)}\}\\ &\quad +\{\|D^2\boldsymbol{G}\|_{\sigma^2}+\|\nabla\lambda\|_{\sigma^2}\}\Big)+\frac{C}{K^2}\|\nabla(\boldsymbol{G}-\boldsymbol{G}^h)\|_{\sigma^2}.\end{aligned}$$

PROOF. Since $\|\nabla(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})\|_{\boldsymbol{L}^2(\Omega)}$ can be bounded as

$$\begin{aligned}\|\nabla(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})\|^2_{\boldsymbol{L}^2(\Omega)} &= \int_\Omega\sigma^2\sigma^{-2}|\nabla(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})|^2\,d\boldsymbol{x}\\ &\leq Ch^2\int_\Omega\sigma^{-2}|\nabla(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})|^2\,d\boldsymbol{x}\leq Ch^2\|\nabla(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})\|^2_{\sigma^{-2}},\end{aligned}$$

the result follows by using Lemma 4.

4. An optimal control problem

This section is devoted to the study of an optimal control problem, where the main goal is to reduce the pressure of an incompressible fluid flow at certain points of a given domain, where the fluid is governed by the generalized Oseen equations. Hence, our main objective is to minimize the pressure at given points of the domain using the action of external given forces with variable amplitudes.

In the optimal control setting, we consider that the total force acting on the fluid $\boldsymbol{f}$ can be decomposed into contributions coming from the usual external force acting on the fluid $\boldsymbol{f}_e$ and a control force $\boldsymbol{F}_c=(\boldsymbol{f}_1,...,\boldsymbol{f}_N)$, where each component $\boldsymbol{f}_i$ has a different amplitude z_i^c.

We are interested in the solution of the following active optimal control problem:

$$\min_{\boldsymbol{z}_c\in\mathbb{R}^N}\frac{1}{2}\|\boldsymbol{z}_o-\boldsymbol{z}_d\|^2_{\mathbb{R}^M}+\frac{\eta}{2}\|\boldsymbol{z}_c\|^2_{\mathbb{R}^N}, \tag{49}$$

subject to

$$\left\{\begin{aligned}-\Delta\boldsymbol{u}+\boldsymbol{b}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}+\nabla p &= \boldsymbol{f}_e+\boldsymbol{z}_c\cdot\boldsymbol{F}_c && \text{in }\Omega,\\ \mathbf{div}\,\boldsymbol{u} &= 0 && \text{in }\Omega,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on }\partial\Omega,\end{aligned}\right. \tag{50}$$

Here, the control variable $\boldsymbol{z}_c = (z_1^c, \ldots, z_N^c) \in \mathbb{R}^N$ represents the amplitude of the control forces. The observation $\boldsymbol{z}_o$ correspond to the set of the values of the pressure field given in M sensors, located in the different points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M \in \Omega$, i.e.

$$\boldsymbol{z}_o = \left(p\left([1\ \boldsymbol{z}_c], \boldsymbol{x}_1\right), ..., p\left([1\ \boldsymbol{z}_c], \boldsymbol{x}_M\right)\right) \in \mathbb{R}^M,$$

where $p([a\ \boldsymbol{z}_c], \boldsymbol{x}_i)$, with $a \in \mathbb{R}$, is the fluid pressure field being the solution of the state equation with right hand side $a\boldsymbol{f}_e + \boldsymbol{z}_c \cdot \boldsymbol{F}_c$ evaluated at the point $\boldsymbol{x}_i \in \Omega$. Finally, $\boldsymbol{z}_d$ represents the desire state, that in our case will be considered to be $\boldsymbol{z}_d = \boldsymbol{0}$, and $\eta > 0$ represents the cost of the control.

Notice that, from the linearity of the state equation, we can write the solution of the state equation as

$$p = p_e + \sum_{k=1}^{N} z_k^c p_k,$$

where p_e and p_k are the solutions of the state equation with right hand sides $\boldsymbol{f}_e$ and $\boldsymbol{f}_k$, respectively. In order to compute the numerical approximation of the optimal control, i.e., $\boldsymbol{z}_{h,c}$ we use the equivalent formulation of the reduced optimization scheme proposed in [2] and analyzed in [3], where the necessary and sufficient condition is given by the Euler equation, which in our framework results in finding the solution of the following linear system:

$$(\boldsymbol{Q}_h + \eta \boldsymbol{I})\boldsymbol{z}_{h,c} = -\boldsymbol{d}_h,$$

where each entry of the matrix $\boldsymbol{Q}_h \in \mathbb{R}^{N\times N}$ and the vector $\boldsymbol{d}_h \in \mathbb{R}^N$ are given by

$$(\boldsymbol{Q}_h)_{ij} = \langle \boldsymbol{q}_{h,i}, \boldsymbol{q}_{h,j} \rangle_{\mathbb{R}^M} \quad \text{and} \quad (\boldsymbol{d}_h)_i = \langle \boldsymbol{q}_{h,e}, \boldsymbol{q}_{h,j} \rangle_{\mathbb{R}^M},$$

with

$$\boldsymbol{q}_{h,e} = (p_h([1\ \boldsymbol{0}], \boldsymbol{x}_1), ..., p_h([1\ \boldsymbol{0}], \boldsymbol{x}_M))$$

and

$$\boldsymbol{q}_{h,k} = (p_h([0\ \boldsymbol{e}_k], \boldsymbol{x}_1), ..., p_h([0\ \boldsymbol{e}_k], \boldsymbol{x}_M)),$$

where $\boldsymbol{e}_k$, is the vector of the canonical basis of $\mathbb{R}^N$ and p_h is the finite element approximation of the pressure field of the state equation. As we mentioned, the previous optimal control problem fits the framework analyzed in [3], and so under the assumptions of Theorem 1, using Theorem 3.9 combined with (3.9) and (3.10) from the cited reference, and the error for the pressure variable (25), the following result follows.

Theorem 3. *Let $\boldsymbol{z}_{h,c}$ the approximation of the optimal control solution $\boldsymbol{z}_c$ of* (49)-(50). *Then, there exists a positive constant C such that*

$$\|\boldsymbol{z}_c - \boldsymbol{z}_{h,c}\|_{\mathbb{R}^N} \le C|\log h|^{1/2} \left(\|\nabla(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{\boldsymbol{L}^\infty(\Omega)} + \|p - P_h p\|_{L^\infty(\Omega)}\right).$$

Clearly, the rate of convergence of the previous result will depend on the general a priori regularity result for the continuous solution (see [23, 24, 18]). Assuming that $(\boldsymbol{u}, p) \in \boldsymbol{W}^{k+1,\infty}(\Omega) \times W^{k,\infty}(\Omega)$ where Ω is a convex polygonal domain, we will have that

$$\|\boldsymbol{z}_c - \boldsymbol{z}_{h,c}\|_{\mathbb{R}^N} \le h^k |\log h|^{1/2} \left(\|\boldsymbol{u}\|_{\boldsymbol{W}^{k+1,\infty}(\Omega)} + \|p\|_{W^{k,\infty}(\Omega)}\right).$$

We can avoid having to check such a requirement by assuming that $\partial\Omega \in \mathcal{C}^{k+1}$ and $\boldsymbol{f} \in \boldsymbol{W}^{k-1,\infty}$, which leads to the convergence result

$$\|\boldsymbol{z}_c - \boldsymbol{z}_{h,c}\|_{\mathbb{R}^N} \le Ch^k |\log\ h|^{3/2}, \tag{51}$$

upon using a priori estimates and standard interpolation error estimates.

5. Numerical experiments

In this section we study the approximation of an optimal control problem based on the algorithm for the approximation of the optimal control problem presented in Table 2 (see the Appendix). In the analysis we denote by $Ndof$ the number of degrees of freedom and we take $\Omega = (0,1) \times (0,1)$.

Example: The control forces $\boldsymbol{f}_i$ are built using the following functions which are solution to (1):

$$
\begin{aligned}
p_1(x,y) &= xye^{x-y} + 2e^{-1} - 1, & \boldsymbol{u}_1(x,y) &= \left[y^3 - y^5, x^3 - x^5\right], \\
p_2(x,y) &= xye^{x} - \frac{1}{2}, & \boldsymbol{u}_2(x,y) &= \left[e^y + \cos(y), \cos(x) - e^x\right], \\
p_3(x,y) &= xye^{y-x} + 2e^{-1} - 1, & \boldsymbol{u}_3(x,y) &= \left[y\sin(y)\cos(y), xe^{-x}\sin(x)\right],
\end{aligned}
$$

The external force $\boldsymbol{f}_e$, is built by considering the following solution of (1) given by

$$
p_0(x,y) = xy(x^2+1)(y^2+1) - \frac{9}{16} \quad \text{and} \quad \boldsymbol{u}_0(x,y) = \left[y - \frac{(1-e^{y/\nu})}{(1-e^{1/\nu})}, x - \frac{(1-e^{x/\nu})}{(1-e^{1/\nu})}\right],
$$

with the vector field given by $\boldsymbol{b} = [y\cos(y), x\sin(x)]$. We take the fluid viscosity as $\nu = 1$, $\nu = 0.01$ and $\nu = 0.005$, $\kappa = 1$ and as cost of the control $\eta = 0$, $\eta = 1$ and $\eta = 0.0001$, and in Table 1 we present the location of the sensors used in the optimal control problem.

Location of the sensors						
x_i	0.75	0.75	0.75	0.25	0.25	0.25
y_i	0.25	0.50	0.75	0.25	0.50	0.75

Table 1: Location of the sensors for the example.

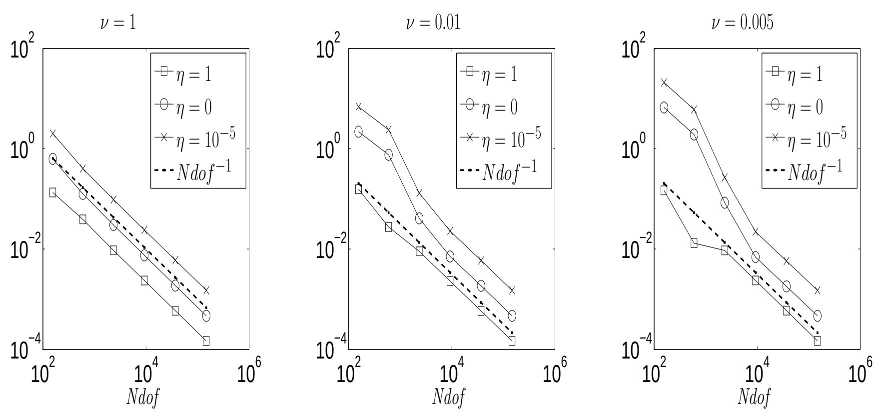

Figure 1: Convergence history of $\|\boldsymbol{z}_c - \boldsymbol{z}_{h,c}\|_{\boldsymbol{L}^\infty(\Omega)}$, for the approximated optimal control.

From Figure 1, we can see that the convergence rate for the approximated control variable is 2, presenting a super convergence result, since even assuming optimal convergence of the pressure field, that is, dropping the $|\log h|^{1/2}$ term in (25), the standard interpolation results will not provide the second order convergence. The previous behaviour was also noticed in [5] where super convergence was attained.

Remark 3. *Notice that all the previous solutions, used to build the right hand sides, do not vanish on the boundary, hence we follow the analysis presented in Section 8* *from [25], where we impose in problem* (1) *that* $\boldsymbol{u} = \boldsymbol{u}_d$ *on* $\partial\Omega$*, where we assume, in order to interpolate the datum, that* $\boldsymbol{u}_d \in \boldsymbol{H}^1(\partial\Omega)$ *which have to satisfies the compatibility condition*

$$\int_{\partial\Omega} \boldsymbol{u}_d \cdot \boldsymbol{n} \; ds = 0,$$

then in [25], to build its discrete counterpart we now let $\mathcal{L}(\boldsymbol{u}_d)$ *be a piecewise quadratic interpolant defined as follows: for an edge* γ *of the partition lying on the boundary* $\partial\Omega$*, with* $\boldsymbol{x}_1$ *and* $\boldsymbol{x}_2$ *being the end points of such a segment,*

$$\mathcal{L}(\boldsymbol{u}_d)\Big|_{\gamma} \quad = \quad \boldsymbol{\alpha}_1\lambda_1 + \boldsymbol{\alpha}_2\lambda_2 + \boldsymbol{\alpha}_3\lambda_1\lambda_2,$$

where λ_i *is the usual linear hat function associated with the vertex* $\boldsymbol{x}_i$*. Taking* $\boldsymbol{\alpha}_i = \boldsymbol{u}_d(\boldsymbol{x}_i)$ *for* $i = 1, 2$ *and* α_3 *being fixed by requiring that*

$$\int_{\partial\Omega} \left(\mathcal{L}(\boldsymbol{u}_d) - \boldsymbol{u}_d\right) \; ds = \mathbf{0},$$

it follows by construction that $\int_{\partial\Omega} \mathcal{L}(\boldsymbol{u}_d) \cdot \boldsymbol{n} \; ds = 0$*.*

6. Appendix

In Table 2 we present the algorithm to approximate the solution of the optimal control problem, which is based on the linearity of the system which allows us to rewrite the cost functional as a quadratic functional leading to the resolution of a quadratic programming problem.

	Algorithm for the approximation of the optimal control problem (49)-(50).
1:	Determine the physical parameters and the external force $\boldsymbol{f}_e$.
2:	Determine the location $\boldsymbol{x}_1, ..., \boldsymbol{x}_M$ of the M sensors inside the domain, also the N control forces $\boldsymbol{f}_1, ..., \boldsymbol{f}_N$, and the cost of the control η.
3:	Build a partition $\mathcal{P}$ of the domain Ω.
4:	Compute the approximated solutions $p_h([1\ \mathbf{0}], \boldsymbol{x})$ and $p_h([0\ \boldsymbol{e}_k], \boldsymbol{x})$ of the following generalized Oseen problem using a Taylor–Hood finite element scheme, based on the partition $\mathcal{P}$: $-\Delta\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{u} + \boldsymbol{u} + \nabla p = \boldsymbol{f}_e + \boldsymbol{z}_c \cdot \boldsymbol{F}_c$ and $\mathbf{div}\ \boldsymbol{u} = 0$ in Ω, $\boldsymbol{u} = \boldsymbol{u}_d$ on $\partial\Omega$.
5:	Compute the matrix $\boldsymbol{Q}_h \in \mathbb{R}^{N\times N}$ and vector $\boldsymbol{d}_h \in \mathbb{R}^N$ given by $(\boldsymbol{Q}_h)_{kj} = \langle \boldsymbol{q}_{h,j}, \boldsymbol{q}_{h,k}\rangle_{\mathbb{R}^M}$ and $(\boldsymbol{d}_h)_j = \langle \boldsymbol{q}_{h,e}, \boldsymbol{q}_{h,j}\rangle_{\mathbb{R}^M}$ for $k, j = 1, \dots, N$, where $\boldsymbol{q}_{h,e} = (p_h([1\ \mathbf{0}], \boldsymbol{x}_1), \dots, p_h([1\ \mathbf{0}], \boldsymbol{x}_M))$, $(\boldsymbol{d}_h)_j = \langle \boldsymbol{q}_{h,e}, \boldsymbol{q}_{h,j}\rangle_{\mathbb{R}^M}$, for $k = 1, \dots, N$.
6:	Compute the solution of the following linear system $(\boldsymbol{Q}_h + \eta\boldsymbol{I})\boldsymbol{z}_{h,c} = -\boldsymbol{d}_h$, to obtain an approximation of the optimal control amplitudes $\boldsymbol{z}_{h,c}$.
7:	Compute the approximated controlled pressure field $p_h(\boldsymbol{x}) = p_h([1\ \mathbf{0}], \boldsymbol{x}) + \sum_{k=1}^{N} z_k^c p_h([0\ \boldsymbol{e}_k], \boldsymbol{x})$.
8:	Refine the partition and go to step 4.

Table 2: Algorithm for the resolution of the optimal control problem.

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